Semigroup Forum (2012) 84:241–266
DOI 10.1007/s00233-011-9337-3

RESEARCH ARTICLE

Plactic algebra of rank 3

Łukasz Kubat · Jan Okniński

Received: 27 May 2011 / Accepted: 16 August 2011 / Published online: 7 September 2011
© The Author(s) 2011. This article is published with open access at Springerlink.com

Abstract The structure of the algebra $K[M]$ of the plactic monoid M of rank 3 over a field K is studied. The minimal prime ideals of $K[M]$ are described. There are only two such ideals and each of them is a principal ideal determined by a homogeneous congruence on M. Moreover, in case K is uncountable and algebraically closed, the left and right primitive spectrum and the corresponding irreducible representations of the algebra $K[M]$ are described. All these representations are monomial. As an application, a new proof of the semiprimitivity of $K[M]$ is given.

Keywords Plactic algebra · Primitive ideals · Simple modules

1 Introduction

For an integer $n \geq 1$ we consider the finitely presented monoid $M_n = \langle a_1, \ldots, a_n \rangle$ defined by the relations

$$a_i a_k a_j = a_k a_i a_j \quad \text{for } i \leq j < k,$$
$$a_j a_i a_k = a_j a_k a_i \quad \text{for } i < j \leq k.$$

It is called the plactic monoid of rank n (cf. [15]). It is known that the elements of M_n can be written in a canonical form, from which it follows that they are in a one-to-one

Communicated by Mohan S. Putcha.

The second author is supported by MNiSW research grant N201 420539 (Poland).

Ł. Kubat
Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-956 Warsaw, Poland
e-mail: L.Kubat@impan.pl

J. Okniński (✉)
Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland
e-mail: Okninski@mimuw.edu.pl

correspondence with Young tableaux of certain type. Because of its strong relations to Young tableaux, the plactic monoid turned out to be very useful in representation theory and algebraic combinatorics. It has already proved to be a classical tool in these theories (cf. [7, 14]). The combinatorics of M_n has been extensively studied but there are only a few preliminary results on the algebraic structure of the monoid algebra $K[M_n]$ of M_n over a field K (cf. [3]). If $n < 3$ then $K[M_n]$ is prime and semiprimitive, and the structure of $K[M_n]$ is pretty well understood. In case $n = 3$, $K[M_n]$ is not prime, but it is still semiprimitive. Moreover, if $n > 3$ then $K[M_n]$ is not semiprime.

The results of this paper contribute to the general program of studying finitely presented algebras defined by homogeneous semigroup presentations. We say that an algebra A with unity is defined by homogeneous semigroup relations if it is given by a presentation $A = \langle X : R\rangle$, where X is a set of free generators of a free algebra over K and R is a set of relations of the form $v = w$, where v, w are words of equal lengths in the generators from X. In this case A may be identified with the semigroup algebra $K[S]$, where S is the monoid defined by the same presentation (cf. [16]). Notice that there is a natural length function on the underlying monoid S. Certain important classes of such algebras, and of the underlying monoids, have been recently considered (cf. [2, 8–11]). Clearly, the plactic algebra $K[M_n]$ is of this type. Also, the related Chinese algebra is defined by semigroup relations of degree 3 (cf. [9, 10]). Algebras corresponding to the set theoretic solutions of the Yang-Baxter equation and a more general class of related algebras are defined by quadratic semigroup relations (cf. [8, 11]).

For certain important constructions of algebras defined by homogeneous semigroup relations it was shown that the minimal prime ideals have a very special form, which proved to have far reaching consequences for the properties of the algebra (cf. [10, 11]). One might expect that this is a more general phenomenon occurring in this class of algebras. If ρ is a congruence on a semigroup S then by I_ρ we denote the ideal of $K[S]$ spanned as a vector space over K by the set $\{v - w : (v, w) \in \rho\}$. Therefore $K[S]/I_\rho \cong K[S/\rho]$. In particular, every minimal prime ideal P of the Chinese algebra $K[S]$ (for the Chinese monoid S) is of the form $P = I_{\rho_P}$ for a homogeneous congruence $\rho_P = \{(v, w) : v - w \in P\}$ on S. The latter means that $(v, w) \in \rho_P$ implies that v, w have equal length in the generators of S. In particular, $K[S]/P \cong K[S/\rho_P]$ is again an algebra defined by homogeneous semigroup relations. One also shows that there are finitely many minimal prime ideals P, each of them is finitely generated and the Jacobson radical of $K[S]$ is nilpotent (cf. [4, 10]). Moreover, there are other results showing that the class of algebras defined by homogeneous semigroup relations has very special properties (cf. [11]). For example, every such right noetherian algebra with finite Gelfand-Kirillov dimension satisfies a polynomial identity. It seems that in the study of algebras of this type also the irreducible representations should play a crucial role.

Our aim is to consider problems of this type for the class of plactic algebras. We establish a remarkable form of minimal prime ideals of the plactic algebra $K[M_3]$ of rank 3 (see Theorem 2.4). There are only two such ideals and each of them is a principal ideal of the form I_ρ for a homogeneous congruence ρ on M_3. Moreover, in case K is uncountable and algebraically closed (e.g., $K = \mathbb{C}$), we describe the left and right primitive spectrum of $K[M_3]$ and the corresponding irreducible representations of $K[M_3]$ (see Theorem 3.5 and Theorem 3.9). In particular, every such representation is monomial.

2 Minimal prime ideals

We start with recalling some basic properties of M_n. It is known that every element of M_n has a unique presentation in the canonical form. Namely, by a row in M_n we mean an element of the form $a_{i_1}\cdots a_{i_r}$, where $r \geq 1$ and $i_1 \leq \cdots \leq i_r$. A column in M_n is defined as an element $a_{j_1}\cdots a_{j_s}$, where $s \geq 1$ and $j_1 > \cdots > j_s$. We say that a row $v = a_{i_1}\cdots a_{i_r}$ dominates a row $w = a_{j_1}\cdots a_{j_s}$ if $r \leq s$ and $i_k > j_k$ for every $k = 1, \ldots, r$. We write $v \rhd w$ in this case. A tableau is a word $w = w_1 \cdots w_t$ such that all w_i are rows and $w_1 \rhd \cdots \rhd w_t$. Then every element $w \in M_n$ is equal in M_n to a unique tableau (cf. [14, 15]). For example

$$w = a_5\ a_3a_4a_4\ a_2a_3a_3a_3\ a_1a_1a_2a_2a_2a_3$$

is a tableau with the subsequent rows

$$w_1 = a_5, \qquad w_2 = a_3a_4a_4, \qquad w_3 = a_2a_3a_3a_3, \qquad w_4 = a_1a_1a_2a_2a_2a_3.$$

Such a tableau can be presented as a planar object

$$w = \begin{array}{|c|c|c|c|c|c|}
\cline{1-1} a_5 \\
\cline{1-3} a_3 & a_4 & a_4 \\
\cline{1-4} a_2 & a_3 & a_3 & a_3 \\
\hline a_1 & a_1 & a_2 & a_2 & a_2 & a_3 \\
\hline
\end{array}$$

Moreover, the subsequent columns of this array are

$$v_1 = a_5a_3a_2a_1, \qquad v_2 = a_4a_3a_1, \qquad v_3 = a_4a_3a_2,$$
$$v_4 = a_3a_2, \qquad v_5 = a_2, \qquad v_6 = a_3,$$

and then we have also $w = v_1 \cdots v_6$ in M_n. So the row reading of the array agrees with the column reading. We call this the canonical form of the element $w \in M_n$. So the elements of M_n are in a one-to-one correspondence with Young tableaux. In particular, M_n has polynomial growth of degree $n(n+1)/2$ (cf. [6, 12]).

The following is an easy consequence of the defining relations and of the canonical form of elements in M_n described above.

Proposition 2.1 (cf. [14, 15]) *Let $z = a_n \cdots a_1 \in M_n$. Then z is a central and regular element of $K[M_n]$. Moreover, if $w \in M_n$ then $w \in zM_n$ if and only if $w = w_na_n \cdots w_1a_1w_0$ for some $w_0, \ldots, w_n \in M_n$.*

Consider the monoid

$$M_n' = M_n/(z=1),$$

i.e., $M_n' = M_n/\rho$, where ρ is the congruence on M_n generated by the pair $(z, 1)$. In particular, $K[M_n'] \cong K[M_n]/(z-1)$. For a column $v = a_{i_1}\cdots a_{i_r} \in M_n$ we define the complement $v^\perp$ of v by

$$v^\perp = \begin{cases} a_{j_1}\cdots a_{j_{n-r}} & \text{for } r<n,\\ 1 & \text{for } r=n,\end{cases}$$

where $j_1 > \cdots > j_{n-r}$ and $\{i_1,\ldots,i_r\}\cup\{j_1,\ldots,j_{n-r}\} = \{1,\ldots,n\}$. Note that $v^\perp$ is a column. Next, if $w = w_1\cdots w_t$ is a tableau presentation of a word $w \in M_n$, where $w_1,\ldots,w_t$ are the subsequent columns, then let $w^\perp = w_t^\perp\cdots w_1^\perp$. Finally, if w' denotes the image of $w\in M_n$ in M_n' then we define $f_1(w') = (w^\perp)'$. Moreover, we can extend f_1 to $K[M_n']$ by linearity.

Proposition 2.2 (cf. [14]) *The map $f_1\colon K[M_n'] \to K[M_n']$ is an antiautomorphism.*

It is easy to check that if $n \geq 3$ then for

$$\sigma = (a_{n-1}\cdots a_1)(a_n\cdots a_2) - (a_n\cdots a_1)(a_{n-1}\cdots a_2), \qquad \pi = a_1a_n - a_na_1,$$

we have $\sigma K[M_n]\pi = 0$. Hence, $K[M_n]$ is not prime (cf. [3]).

Our aim is to consider the case $n=3$. It is then convenient to write $M = M_3$, $M' = M_3'$, and $a = a_1$, $b = a_2$, $c = a_3$. So $M = \langle a,b,c\rangle$ with the convention that $a<b<c$ (when applying the defining relations of M). Moreover, we shall use the same notation for the elements of M' and the elements of $M\setminus zM$, if unambiguous.

The canonical form of an element $w\in M$ looks in this case as follows:

$$w = (cba)^{k_1}(ba)^{k_2}(ca)^{k_3}(cb)^{k_4}a^{k_5}b^{k_6}c^{k_7},$$

where $k_i \geq 0$ such that either $k_4 = 0$ or $k_5 = 0$. Let

$$N_1 = M/(ac = ca), \qquad N_2 = M/(bacb = cbab),$$

i.e., $N_1 = M/\rho_1$ (respectively $N_2 = M/\rho_2$), where ρ_1 (respectively ρ_2) is the congruence on M generated by the pair (ac, ca) (respectively $(bacb, cbab)$). We use the same notation for the elements of N_1 and N_2 as for M, if unambiguous.

Lemma 2.3 *Every element $u \in N_1$ can be uniquely written in the form*

$$u = (cba)^{k_1}(ba)^{k_2}a^{k_3}(cb)^{k_4}b^{k_5}c^{k_6},$$

where $k_i \geq 0$. Moreover, cba is a central and regular element of $K[N_1]$.

Proof Since we have $ac = ca$ in N_1 and in view of the canonical form of elements in $\langle b,c\rangle \cong M_2$, it is easy to see that every element in N_1 can be written in the above form. We claim that u has only one presentation of this form.

First, for any word w in a, b, c define the number $n(w)$ as follows. If w does not contain a subword of the form cw_2bw_1a for some words w_1, w_2 then define $n(w) = 0$. Whereas if w contains a subword of the type mentioned above, then write w in the form $w = v_0aw_0$ for some words v_0, w_0, where the degree of the word v_0 is as small as possible. Then, write $v_0 = v_1bw_1$ for some words v_1, w_1, where w_1 is of minimal possible degree. Finally, write $v_1 = w_3cw_2$ for some words w_2, w_3, where the degree of the word w_2 is as small as possible. Thus we get $w = w_3cw_2bw_1aw_0$. Then, let $\overline{w} = w_3w_2w_1w_0$ and define $n(w) = n(\overline{w}) + 1$. By Proposition 2.1, we know that for every word w, considered as an element of M, the number $n(w)$ is equal to the maximal $n \geq 0$ such that $w \in (cba)^n M$. Hence $n(w)$ is an invariant of the class of the word w in M. We claim that $n(w)$ is also an invariant of the class of w in the monoid N_1. To prove this fact it suffices to show that if a word w' arises from w, by rewriting w using the relation $ac = ca$ once, then $n(w) = n(w')$. Indeed, if $n(w) = 0$ then also $n(w') = 0$. Further, if $n(w) > 0$ then it is easy to see that the word $\overline{w}'$, obtained from w' according to the rule explained above, is actually equal to the word $\overline{w}$. Therefore $n(\overline{w}) = n(\overline{w}')$, so we conclude that

$$n(w) = n(\overline{w}) + 1 = n(\overline{w}') + 1 = n(w'),$$

as desired.

Now, consider $u \in N_1$, written in the form given in the statement of the lemma, and $v = (cba)^{l_1}(ba)^{l_2}a^{l_3}(cb)^{l_4}b^{l_5}c^{l_6} \in N_1$, where $l_i \geq 0$. Suppose that the images of u and v (under the homomorphisms $N_1 \to \langle a, b\rangle$ and $N_1 \to \langle b, c\rangle$ sending c to 1 and a to 1, respectively) are equal in the plactic monoids $\langle a, b\rangle$, $\langle b, c\rangle$ of rank 2, respectively. Looking at these images and using the canonical form of elements in these plactic monoids, we get

$$k_1 + k_2 = l_1 + l_2, \qquad k_4 + k_5 = l_4 + l_5, \qquad k_3 = l_3,$$
$$k_1 + k_4 = l_1 + l_4, \qquad k_2 + k_5 = l_2 + l_5, \qquad k_6 = l_6.$$

Therefore, if $u = v$ in N_1, then $k_1 = n(u) = n(v) = l_1$ and the equalities displayed above give $k_i = l_i$ for all i. This proves the claim. Clearly, the remaining assertion of the lemma now also follows. This completes the proof. □

Our main result in this section reads as follows.

Theorem 2.4 *Let $K[M]$ be the plactic algebra of rank 3 over a field K. Then the principal ideals $P_1 = (ac - ca)$ and $P_2 = (bacb - cbab)$ are the only minimal prime ideals of $K[M]$.*

Proof As noticed above, we have $\sigma K[M]\pi = 0$ for $\sigma = bacb - cbab$ and $\pi = ac - ca$. Thus every prime ideal of $K[M]$ contains P_1 or P_2.

Let $z = cba$. We know that $K[M]\langle z\rangle^{-1} \cong K[M'][x, x^{-1}]$, the Laurent polynomial ring in one variable over $K[M']$, with the isomorphism mapping $w = vz^n \in M\langle z\rangle^{-1}$, where $n \in \mathbb{Z}$ and $v \in M \setminus zM$, to the element $v'x^n \in K[M'][x, x^{-1}]$ with v' denoting the image of v in M' (cf. [3]).

Let us assume, for a moment, that P_1 is a prime ideal of $K[M]$. Because z is a nonzero central element in $K[M]/P_1 \cong K[N_1]$, the central localization

$$K[N_1]\langle z\rangle^{-1} \cong K[M'/(ac=ca)][x,x^{-1}]$$

is prime, so the algebra $K[M'/(ac=ca)]$ is also prime. Applying the antiautomorphism f_1 from Proposition 2.2 we see that $f_1(\pi') = bacb - b = \sigma'$. Therefore the algebra $K[M'/(bacb=cbab)]$ is prime, hence we conclude that

$$K[N_2]\langle z\rangle^{-1} \cong K[M'/(bacb=cbab)][x,x^{-1}]$$

is also prime. Since this is a central localization of the algebra $K[N_2] \cong K[M]/P_2$, the latter must be prime. Therefore P_2 is a prime ideal of $K[M]$. Now, the assertion of the theorem follows by the first paragraph of the proof.

Hence, it suffices to show that $K[N_1]$ is a prime algebra. So assume, on the contrary, that $\zeta K[N_1]\xi = 0$ for some $0 \neq \zeta, \xi \in K[N_1]$. We may assume, replacing ζ and ξ by their certain homogeneous components, that ζ and ξ are homogeneous with respect to every generator a, b, c. Let $\zeta = \sum_{i=1}^{r} \lambda_i w_i$, where $0 \neq \lambda_i \in K$ and $w_i \in N_1$ are pairwise distinct. Let $k_{ij} \geq 0$ denote the subsequent exponents in the canonical form of w_i given in Lemma 2.3. If $k_{i3} \geq 1$ for some i, then

$$cbw_i = (cba)^{k_{i1}+1}(ba)^{k_{i2}}a^{k_{i3}-1}(cb)^{k_{i4}}b^{k_{i5}}c^{k_{i6}}.$$

Moreover, if $k_{i3} = 0$ then the exponent of a in cbw_i is also equal to zero. So, if $k_{i3} \geq 2$ for some i, then $cb\zeta \neq 0$. Hence, replacing ζ by $(cb)^p\zeta$ for some $p \geq 0$, we may assume $k_{i3} \in \{0, 1\}$ for all i. Similarly, if $k_{i2} \geq 1$, then

$$cw_i = (cba)^{k_{i1}+1}(ba)^{k_{i2}-1}a^{k_{i3}}(cb)^{k_{i4}}b^{k_{i5}}c^{k_{i6}}$$

and if $k_{i2} = 0$ then the exponent of ba in cw_i is also equal to zero. Therefore, if $k_{i2} \geq 2$ for some i, then $c\zeta \neq 0$. So, replacing ζ by $c^q\zeta$ for some $q \geq 0$, we reduce to the case where $k_{i2} \in \{0, 1\}$ for all i. Since cba is regular, we may assume also $k_{i1} = 0$ for some i. Since ζ is homogeneous, this implies that $\deg_a \zeta \leq 2$.

Furthermore, we may assume that ζ has the least possible degree among all nonzero homogeneous elements $\zeta' \in K[N_1]$ with the latter property and such that $\zeta' K[N_1]\xi' = 0$ for some $0 \neq \xi' \in K[N_1]$. Similarly, using the same reductions, we can assume the same properties on the element ξ.

First, consider the case where $\deg_a \zeta = 2$. Then ζ has the form

$$\zeta = (cba)^2\zeta_1 + cbaba\zeta_2 + cbaa\zeta_3 + baa\zeta_4,$$

where $\zeta_i \in K[\langle b, c\rangle]$. If $\zeta_4 = 0$, then $\zeta \in cbaK[N_1]$, so canceling cba we get a contradiction with the minimal choice of ζ. Now $cb\zeta = 0$, because if $cb\zeta \neq 0$ then replacing ζ by $cb\zeta$ and canceling cba leads also to a contradiction. Thus

$$0 = cb\zeta = (cba)^2cb\zeta_1 + (cba)^2b\zeta_2 + (cba)^2\zeta_3 + cbaba\zeta_4$$

implies $ba\zeta_4 = 0$, because only $cbaba\zeta_4 \notin (cba)^2K[N_1]$. Hence, we get $0 = cba\zeta_4$ and, in fact, ζ_4 must be equal to zero, a contradiction. Therefore, we have $\deg_a \zeta \leq 1$ and similarly $\deg_a \xi \leq 1$.

In the case where $\deg_a \zeta = 1$, the element ζ has the form

$$\zeta = cba\zeta_1 + ba\zeta_2 + a\zeta_3,$$

where $\zeta_i \in K[\langle b, c\rangle]$. Consider

$$c\zeta = cbac\zeta_1 + cba\zeta_2 + ac\zeta_3.$$

In case $c\zeta \neq 0$, we may replace ζ by $c\zeta$, so we may assume $\zeta_2 = 0$. Notice that we increased the degree of ζ, so we cannot use the minimality of ζ anymore. However, if the degree of an element obtained from the original ζ, by some further multiplications, does not exceed $\deg \zeta + 2$, then canceling cba, if possible, leads also to a contradiction. So, in case $\zeta_2 = 0$, we get $\zeta = cba\zeta_1 + a\zeta_3$ and $b\zeta = cbab\zeta_1 + ba\zeta_3$. Moreover, if $b\zeta = 0$, then $\zeta_3 = 0$ and $\zeta \in cbaK[N_1]$, a contradiction. Thus we have $b\zeta \neq 0$, and replacing ζ by $b\zeta$ we may, therefore, assume $\zeta_3 = 0$, i.e., $\zeta = cba\zeta_1 + ba\zeta_2$. Now, consider $c\zeta = cbac\zeta_1 + cba\zeta_2$. If $c\zeta \neq 0$, then canceling cba we reduce to the case $\zeta \in K[\langle b, c\rangle]$. However, if $c\zeta = 0$, then we get $c\zeta_1 + \zeta_2 = 0$ and $\zeta_3 = 0$.

Concluding, it suffices to consider the case where $\zeta \in K[\langle b, c\rangle]$, or $c\zeta_1 + \zeta_2 = 0$ and $c\zeta_3 = 0$. The former case will be considered later. So assume the latter case and consider

$$b\zeta = cbab\zeta_1 + bab\zeta_2 + ba\zeta_3.$$

In case $b\zeta = 0$, we get $cbab\zeta_1 = 0$ and $bab\zeta_2 + ba\zeta_3 = 0$. Canceling cba, we get $b\zeta_1 = 0$, hence $cb\zeta_1 = 0$. Since cb is a regular element of $K[\langle b, c\rangle] \cong K[M_2]$, we conclude that $\zeta_1 = 0$. Similarly,

$$0 = c(bab\zeta_2 + ba\zeta_3) = cba(b\zeta_2 + \zeta_3)$$

implies $b\zeta_2 + \zeta_3 = 0$. Thus we have $\zeta = (ba - ab)\zeta_2$. This case will be considered later.

In case $b\zeta \neq 0$, we may replace ζ by $b\zeta$, so we may assume that $\zeta_3 = 0$. Therefore, ζ has the form $\zeta = cba\zeta_1 + ba\zeta_2$. Next, consider $cb\zeta = cbacb\zeta_1 + cbab\zeta_2$. If $cb\zeta \neq 0$, then replacing ζ by $cb\zeta$ and canceling cba, we reduce to the case $\zeta \in K[\langle b, c\rangle]$. Whereas, if $cb\zeta = 0$, then we get $cb\zeta_1 + b\zeta_2 = 0$. In this situation, let

$$\zeta_1 = \sum_{i=1}^{s} \lambda_i (cb)^{k_{i1}} b^{k_{i2}} c^{k_{i3}}, \qquad \zeta_2 = -\sum_{i=1}^{s} \mu_i (cb)^{l_{i1}} b^{l_{i2}} c^{l_{i3}},$$

where $\lambda_i, \mu_i \in K$ and $k_{ij}, l_{ij} \geq 0$ are such that the triples (k_{i1}, k_{i2}, k_{i3}) (respectively (l_{i1}, l_{i2}, l_{i3})) are pairwise distinct. Then $cb\zeta_1 + b\zeta_2 = 0$ yields

$$\sum_{i=1}^{s} \lambda_i (cb)^{k_{i1}+1} b^{k_{i2}} c^{k_{i3}} = \sum_{i=1}^{s} \mu_i (cb)^{l_{i1}} b^{l_{i2}+1} c^{l_{i3}}.$$

Hence, by Lemma 2.3, we conclude that $\lambda_i = \mu_i$ and $l_{i1} = k_{i1} + 1$, $k_{i2} = l_{i2} + 1$, $k_{i3} = l_{i3}$ for all i. Therefore

$$\zeta = cba\zeta_1 + ba\zeta_2 = cba\sum_{i=1}^{s}\lambda_i (cb)^{k_{i1}} b^{l_{i2}+1} c^{k_{i3}} - ba\sum_{i=1}^{s}\lambda_i (cb)^{k_{i1}+1} b^{l_{i2}} c^{k_{i3}}.$$

If $k_{i1} \geq 1$ for all i, then

$$\begin{aligned}\zeta a &= cba\zeta_1 a + ba\zeta_2 a\\ &= cba\sum_{i=1}^{s}\lambda_i (cb)^{k_{i1}} bab^{l_{i2}} c^{k_{i3}} - ba\sum_{i=1}^{s}\lambda_i b^{l_{i2}} (cb)^{k_{i1}+1} ac^{k_{i3}}\\ &= (cba)^2\sum_{i=1}^{s}\lambda_i (cb)^{k_{i1}-1} b^{l_{i2}+1} c^{k_{i3}} - cbaba\sum_{i=1}^{s}\lambda_i (cb)^{k_{i1}} b^{l_{i2}} c^{k_{i3}},\end{aligned}$$

so we can replace ζ by ζa and cancel cba. This implies that we may assume $k_{i1} = 0$ for some i. Therefore, let $k_{11} = 0$. Then $k_{13} + 1 = \deg_c \zeta$, because ζ is homogeneous with respect to c. Thus replacing ζ by $\zeta(ba)^{k_{13}-1} \neq 0$ and canceling $(cba)^{k_{13}-1}$, we reduce to the case where $\deg_c \zeta_1 \leq 1$. This implies that $k_{i1} + k_{i3} \leq 1$ for all i. Therefore, if $k_{13} = 0$, then $\zeta_1 = \lambda b^{m+1}$ and $\zeta_2 = -\lambda cbb^m$ for some $0 \neq \lambda \in K$ and $m \geq 0$ (in fact, $\lambda = \lambda_1$ and $m = l_{12}$). This yields

$$\zeta = cba\zeta_1 + ba\zeta_2 = \lambda(cbab - bacb)b^m.$$

Similarly, if $k_{13} = 1$, then $\zeta_1 = \lambda b^{m+2}c - \mu cbb^{m+1}$ and $\zeta_2 = -\lambda cbb^{m+1}c + \mu(cb)^2 b^m$ for some $\lambda, \mu \in K$ and $m \geq 0$ (in fact, $\lambda = \lambda_1$, $\mu = -\lambda_2$ and $m = l_{22}$). Moreover, if $\mu = 0$, then $m = -1$ is allowed. Therefore, this yields

$$\begin{aligned}\zeta &= cba\zeta_1 + ba\zeta_2\\ &= cba(\lambda b^{m+2}c - \mu cbb^{m+1}) - ba(\lambda cbb^{m+1}c - \mu(cb)^2 b^m)\\ &= \lambda(cbab - bacb)b^{m+1}c - \mu(cbab - bacb)b^m cb\\ &= (cbab - bacb)b^m(\lambda bc - \mu cb).\end{aligned}$$

Thus we have

$$\zeta ba = (\lambda - \mu)(cbab - bacb)b^{m+1}cba.$$

If $\zeta ba \neq 0$, then we may replace ζ by ζba and cancel cba. This gives another reduction to $\zeta = (cbab - bacb)b^{m+1}$. However, if $\zeta ba = 0$, then we get $\lambda = \mu \neq 0$, so we may assume that $\zeta = (cbab - bacb)b^m(bc - cb)$.

Now, let us go back to the cases left earlier. Let $\zeta = (ba - ab)\eta$ for some $\eta \in K[\langle b, c\rangle]$, homogeneous with respect to b and c. Then replacing η by ηb^t for some $t \geq 0$, we reduce to the case $\eta = \lambda(cb)^{n+1}b^m - \mu(cb)^n b^{m+1}c$ for some $\lambda, \mu \in K$ and $n, m \geq 0$. Moreover, if $\lambda = 0$, then $m = -1$ is allowed and similarly, if $\mu = 0$, then

$n = -1$ is allowed. Then $\eta = b^m(\lambda cb - \mu bc)(cb)^n$. Next, replacing η by ηa^n and canceling $(cba)^n$, we may assume that $\eta = b^m(\lambda cb - \mu bc)$. In this case, we get

$$\begin{aligned}\zeta ba &= (ba - ab)\eta ba\\ &= (ba - ab)b^m(\lambda cb - \mu bc)ba\\ &= (\lambda - \mu)(ba - ab)b^{m+1}cba.\end{aligned}$$

If $\zeta ba \neq 0$, then we may replace ζ by ζba and cancel cba. This gives another reduction to $\zeta = (ba - ab)b^{m+1}$. Furthermore, $c^{m+1}\zeta a^{m+1} = (ba - ab)(cba)^{m+1}$, so we can cancel $(cba)^{m+1}$ and we may, therefore, assume that $\zeta = ba - ab$. However, if $\zeta ba = 0$, then we get $\lambda = \mu \neq 0$, so we may assume that $\zeta = (ba - ab)b^m(cb - bc)$. The last case to consider is when $\zeta \in K[\langle b, c\rangle]$. Then, as before, we reduce to the case $\zeta = b^m(\lambda cb - \mu bc)$ for some $\lambda, \mu \in K$ and $m \geq 0$. Therefore

$$\zeta ba = b^m(\lambda cb - \mu bc)ba = (\lambda - \mu)b^{m+1}cba.$$

If $\zeta ba \neq 0$, then canceling cba we may assume that $\zeta = b^{m+1}$ and, in fact, replacing ζ by $c^{m+1}\zeta a^{m+1}$ and canceling $(cba)^{m+1}$ gives $\zeta = 1$. However, if $\zeta ba = 0$, then we have $\lambda = \mu \neq 0$, so we may assume that $\zeta = b^m(cb - bc)$. Therefore, replacing ζ by $c^m\zeta a^m = (cb - bc)(cba)^m$ and canceling $(cba)^m$ gives $\zeta = cb - bc$.

Similarly, using the same reductions, we can assume the same possible forms of the element ξ, namely:

$$\begin{array}{lll} ab - ba, & (ab - ba)b^n, & (ab - ba)b^n(bc - cb),\\ bc - cb, & (bacb - cbab)b^n, & (bacb - cbab)b^n(bc - cb)\end{array}$$

for $n \geq 0$. Hence, it suffices to consider the following cases:

$$(ab - ba)b^n(bc - cb)\ K[N_1]\ (ab - ba)b^m(bc - cb) = 0, \tag{1}$$

$$(ab - ba)b^n(bc - cb)\ K[N_1]\ (bacb - cbab)b^m(bc - cb) = 0, \tag{2}$$

$$(bacb - cbab)b^n(bc - cb)\ K[N_1]\ (ab - ba)b^m(bc - cb) = 0, \tag{3}$$

$$(bacb - cbab)b^n(bc - cb)\ K[N_1]\ (bacb - cbab)b^m(bc - cb) = 0 \tag{4}$$

for $n, m \geq 0$. Applying case (1) to the element $1 \in N_1$, we get an element which is nonzero, as it is easy to see that it has four distinct elements of $cbaN_1$ in its support. In case (2), using the element $a \in N_1$, we get a nonzero element, because it has four distinct elements of $(cba)^2N_1$ in its support. In case (3), substituting $1 \in N_1$, we get again a nonzero element, because it has two distinct elements of $(cba)^2N_1$ in its support. Finally, in case (4), we substitute $a \in N_1$ and we obtain a nonzero element, because it has two distinct elements of $(cba)^3N_1$ in its support. Hence, all cases lead to a contradiction. We have thus shown that $K[N_1]$ is prime. This completes the proof of the theorem. □

It is now possible to give a simple proof of the fact that the algebra $K[M]$ is semiprime (cf. [3]).

Proposition 2.5 *Let P_1 and P_2 be the minimal prime ideals of the plactic algebra $K[M]$ of rank 3 over a field K. Then $P_1 \cap P_2 = 0$.*

Proof Let $\xi = \sum_{i=1}^{n} \lambda_i v_i (bacb - cbab) w_i \in P_1 \cap P_2$, where $\lambda_i \in K$ and $v_i, w_i \in M$. Let us write v_i and w_i in their canonical forms

$$v_i = (cba)^{k_{i1}} (ba)^{k_{i2}} (ca)^{k_{i3}} (cb)^{k_{i4}} a^{k_{i5}} b^{k_{i6}} c^{k_{i7}},$$

$$w_i = (cba)^{l_{i1}} (ba)^{l_{i2}} (ca)^{l_{i3}} (cb)^{l_{i4}} a^{l_{i5}} b^{l_{i6}} c^{l_{i7}},$$

where $k_{ij}, l_{ij} \geq 0$ and $k_{i4}k_{i5} = l_{i4}l_{i5} = 0$ for all i. Firstly, notice that cba and b commute with $bacb - cbab$. Therefore, using $u(bacb - cbab) = 0$ for $u \in \{c, cb, ca\}$, we may assume $k_{i3} = k_{i4} = k_{i7} = 0$ for all i. Similarly, using $(bacb - cbab)u = 0$ for $u \in \{ba, ca, a\}$, we may assume $l_{i1} = l_{i2} = l_{i3} = l_{i5} = 0$ for all i. Thus ξ has the form

$$\begin{aligned}
\xi &= \sum_{i=1}^{n} \lambda_i (cba)^{k_{i1}} (ba)^{k_{i2}} a^{k_{i5}} b^{k_{i6}} (bacb - cbab)(cb)^{l_{i4}} b^{l_{i6}} c^{l_{i7}} \\
&= \sum_{i=1}^{n} \lambda_i (cba)^{k_{i1}} (ba)^{k_{i2}+1} a^{k_{i5}} (cb)^{l_{i4}+1} b^{k_{i6}+l_{i6}} c^{l_{i7}} \\
&\quad - \sum_{i=1}^{n} \lambda_i (cba)^{k_{i1}+1} (ba)^{k_{i2}} a^{k_{i5}} (cb)^{l_{i4}} b^{k_{i6}+l_{i6}+1} c^{l_{i7}}.
\end{aligned}$$

We know that the image of ξ in $K[M]/P_1 \cong K[N_1]$ is equal to zero. Moreover, the elements listed in Lemma 2.3 constitute a basis of $K[N_1]$ over K. Thus we get $\lambda_i = 0$ for all i. The assertion follows. □

Proposition 2.5 implies that $K[M]$ is a subdirect product of $K[N_1]$ and $K[N_2]$, so in particular the monoid M embeds into $N_1 \times N_2$.

Prime ideals of $K[M]$ not intersecting M lead to an interesting class of simple monoids. Recall that $\mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}$ stand for Green's relations (cf. [5]).

Proposition 2.6 *Let $K[M]$ be the plactic algebra of rank 3 over a field K. Let P be a prime ideal of $K[M]$ such that $P \cap M = \emptyset$. Let $z = cba \in M$ and let M_P be the image of M in $K[M]/P$. Then*:

(1) $M_P\langle z\rangle^{-1}$ *is a simple monoid,*
(2) $M_{P_1}\langle z\rangle^{-1} \cong M_{P_2}\langle z\rangle^{-1}$.

Moreover, if $P = P_1$ and

$$v = (cba)^{k_1} (ba)^{k_2} a^{k_3} (cb)^{k_4} b^{k_5} c^{k_6}, \qquad w = (cba)^{l_1} (ba)^{l_2} a^{l_3} (cb)^{l_4} b^{l_5} c^{l_6}$$

are elements of $M_P\langle z\rangle^{-1}$, where $k_i, l_i \in \mathbb{Z}$ and $k_j, l_j \geq 0$ for $j \geq 2$, then:

(3) $U = \langle z, z^{-1}\rangle$ *is the group of units of $M_P\langle z\rangle^{-1}$,*

(4) $v\mathcal{L}w$ *in* $M_P\langle z\rangle^{-1}$ *if and only if* $k_i = l_i$ *for* $i = 4, 5, 6$,
(5) $v\mathcal{R}w$ *in* $M_P\langle z\rangle^{-1}$ *if and only if* $k_i = l_i$ *for* $i = 2, 3, 5$,
(6) $v\mathcal{H}w$ *in* $M_P\langle z\rangle^{-1}$ *if and only if* $k_i = l_i$ *for* $i = 2, 3, 4, 5, 6$,
(7) $v\mathcal{D}w$ *in* $M_P\langle z\rangle^{-1}$ *if and only if* $k_5 = l_5$.

Proof By Lemma 2.3 z is a nonzero central element of $M_P \subseteq K[M]/P$, so $M_P\langle z\rangle^{-1}$ is a submonoid of the central localization $(K[M]/P)\langle z\rangle^{-1}$. From Proposition 2.1 it easily follows that if $u \in M$ then there exist $x, y \in M$ such that $xuy \in \langle z\rangle$. This implies that $M_P\langle z\rangle^{-1}$ is simple, hence (1) follows.

From the proof of Theorem 2.4 we know that $M_{P_1}\langle z\rangle^{-1} \cong \mathbb{Z} \times M'/(ac = ca)$ and $M_{P_2}\langle z\rangle^{-1} \cong \mathbb{Z} \times M'/(bacb = cbab)$. Moreover, the antiautomorphism f_1 from Proposition 2.2 yields an isomorphism $M'/(ac = ca) \cong M'/(bacb = cbab)$. Hence (2) follows.

Assume that $P = P_1$ and $v, w \in M_P\langle z\rangle^{-1}$ are as above. Suppose $v\mathcal{L}w$. If $u \in M_P\langle z\rangle^{-1}$ then it is easy to see that the exponent of cb in the canonical form of uv is $\geq k_4$. Therefore, if $v\mathcal{L}w$ then $k_4 = l_4$. Similarly, one shows that $k_6 = l_6$. Then it also easily follows that $k_5 = l_5$. The converse is clear. This implies that (4) holds. The proof of assertion (5) is symmetric. Then (6) and assertion (3) are immediate consequences and (7) also follows. □

It is natural to ask whether the simple monoids constructed above have been known before.

3 Irreducible representations and primitive spectrum

Throughout this section we shall assume that the field K is uncountable and algebraically closed (for convenience one may take $K = \mathbb{C}$). It is worth mentioning that some of the results obtained below do not depend on this assumption, but this assumption is crucial for the main results of this section.

As a first observation let us note that $K[M]$ does not have finite-dimensional irreducible representations of dimension greater than 1.

Proposition 3.1 *Every finite-dimensional irreducible representation of* $K[M]$ *is* 1-*dimensional.*

Proof Let $\varrho\colon K[M] \to \mathrm{M}_n(K)$ be such a representation. If we have $\varrho(cba) \neq 0$ then, since cba is central in $K[M]$, we get $\varrho(cba) = \lambda$ (a nonzero scalar matrix) for some $0 \neq \lambda \in K$. Indeed, the K-subalgebra $\varrho(K[M])$ of $\mathrm{M}_n(K)$ is simple, because it is primitive and finite-dimensional, so $\varrho(cba)$ is invertible in it, whence in $\mathrm{M}_n(K)$. This implies that $\varrho(a), \varrho(b), \varrho(c)$ are invertible matrices. Now, applying the relation $aba = baa$ in M, we get $\varrho(aba) = \varrho(baa)$. So, canceling $\varrho(a)$, we get $\varrho(ab) = \varrho(ba)$. Similarly $\varrho(ac) = \varrho(ca)$ and $\varrho(bc) = \varrho(cb)$. It follows that $\varrho(M)$ is commutative, hence we get $n = 1$. In the case where $\varrho(cba) = 0$ one can show (see Theorem 3.5) that $\varrho(a) = 0$ or $\varrho(b) = 0$ or $\varrho(c) = 0$. So, without loss of generality, we may assume $\varrho(c) = 0$. Moreover, if $\varrho(ba) = 0$, then we have $\varrho(a) = 0$ or $\varrho(b) = 0$

(see Theorem 3.5 again). Hence $\varrho(K[M])$ is commutative, so this forces $n = 1$. Finally, if $\varrho(ba) \neq 0$, then $\varrho(a)$ and $\varrho(b)$ are invertible, because $\varrho(ba)$ is central in $\varrho(K[M])$. Thus we conclude that $\varrho(K[M])$ is again commutative, so $n = 1$. □

Because we know that the algebra $K[M]$ has an irreducible representation of dimension exceeding 1, Proposition 3.1 assures that such a representation must be infinite-dimensional. Our aim is to describe all such representations and the corresponding primitive ideals of $K[M]$.

We start with the following well-known result.

Proposition 3.2 (cf. [3, 13]) *Let F be an algebraically closed field. Let A be a left or right primitive F-algebra such that $\dim_F A < |F|$. Then A is a central F-algebra.*

Recall that the bicyclic monoid is the monoid given by the finite presentation

$$B = \langle p, q : pq = 1\rangle.$$

It is easy to see that $K[B] \cong K\langle x, y\rangle/(xy - 1)$, where $K\langle x, y\rangle$ is the free algebra in two variables over K (cf. [13]) or more generally $K[B] \cong K\langle x, y\rangle/(xy - \delta)$ for every $0 \neq \delta \in K$. Now, we describe the primitive spectrum of $K[B]$.

Proposition 3.3 (cf. [3]) *Let P be a left or right primitive ideal of $K[B]$. Then $P = 0$ or $P = (p - \alpha, q - \beta)$ for some $\alpha, \beta \in K$ with $\alpha\beta = 1$.*

Proof It is well known that $K[B]$ is a left and right primitive ring (cf. [13]). So, let $0 \neq \xi = \sum_{i,j=0}^{n} \lambda_{ij} q^i p^j \in P$, where $\lambda_{ij} \in K$. If $i_0 = \min\{i : \lambda_{ij} \neq 0 \text{ for some } j\}$ and $j_0 = \min\{j : \lambda_{i_0 j} \neq 0\}$ then we get

$$\begin{aligned}(1 - qp)p^{i_0}\xi q^{j_0}(1 - qp) &= \sum_{i,j=0}^{n} \lambda_{ij}(1 - qp)p^{i_0}q^i p^j q^{j_0}(1 - qp)\\ &= \sum_{j=0}^{n} \lambda_{i_0 j}(1 - qp)p^j q^{j_0}(1 - qp) = \lambda_{i_0 j_0}(1 - qp),\end{aligned}$$

because $1 - qp$ is an idempotent. Hence $1 - qp \in P$ and P corresponds to a left or right primitive ideal of $K[B]/(1 - qp) \cong K[x, x^{-1}]$, the Laurent polynomial ring in one variable over K. Thus $P = (p - \alpha, q - \beta)$ for some $\alpha, \beta \in K$ with $\alpha\beta = 1$, and the assertion follows. □

Certain natural antiendomorphisms of $K[M]$, some of them generalizing the antiautomorphism f_1 from Proposition 2.2, will be very useful.

Lemma 3.4 *There exists an involution $g\colon K[M] \to K[M]$ such that*

$$g(a) = c, \qquad g(b) = b, \qquad g(c) = a.$$

Moreover, for $0 \neq \lambda \in K$, *there exists an antimonomorphism* $f_\lambda \colon K[M] \to K[M]$ *satisfying*

$$f_\lambda(a) = \lambda^{-1}cb, \qquad f_\lambda(b) = ca, \qquad f_\lambda(c) = ba,$$

which induces an involution of $K[M]/(cba - \lambda)$.

Proof Let $K\langle x, y, z\rangle$ be the free algebra in three variables over K. First, we can define antihomomorphisms $g \colon K\langle x, y, z\rangle \to K[M]$ and $f_\lambda \colon K\langle x, y, z\rangle \to K[M]$ on generators as follows:

$$g(x) = c, \qquad g(y) = b, \qquad g(z) = a,$$
$$f_\lambda(x) = \lambda^{-1}cb, \qquad f_\lambda(y) = ca, \qquad f_\lambda(z) = ba.$$

Then it is easy to see that g and f_λ respect the defining relations of M. Indeed, for example

$$\begin{aligned} g(yxz) = g(z)g(x)g(y) &= acb \\ &= cab = g(x)g(z)g(y) = g(yzx), \\ f_\lambda(yxz) = f_\lambda(z)f_\lambda(x)f_\lambda(y) &= \lambda^{-1}bacbca \\ &= \lambda^{-1}cbbaca = f_\lambda(x)f_\lambda(z)f_\lambda(y) = f_\lambda(yzx). \end{aligned}$$

This implies that the defined antihomomorphisms induce antiendomorphisms of $K[M]$, also denoted by g and f_λ, respectively. Considering g in this way, it is obvious that g is an involution of $K[M]$. Furthermore

$$f_\lambda(f_\lambda(a)) = f_\lambda(\lambda^{-1}cb) = \lambda^{-1}f_\lambda(b)f_\lambda(c) = \lambda^{-1}caba = \lambda^{-1}(cba)a,$$

and similarly $f_\lambda(f_\lambda(w)) = \lambda^{-1}(cba)w$ for $w \in \{b, c\}$. Since cba is regular in $K[M]$, it follows that f_λ is an antimonomorphism. Moreover, we have

$$f_\lambda(cba) = f_\lambda(a)f_\lambda(b)f_\lambda(c) = \lambda^{-1}cbcaba = \lambda^{-1}(cba)^2.$$

This means that f_λ may be viewed as an antiendomorphism of $K[M]/(cba - \lambda)$. Considering f_λ in this way, we get $f_\lambda(f_\lambda(w)) = w$ for $w \in \{a, b, c\}$, which assures that f_λ is an involution of $K[M]/(cba - \lambda)$. This completes the proof. □

The involution g gives a one-to-one correspondence between left and right primitive ideals of $K[M]$. Similarly, for $0 \neq \lambda \in K$, the involution induced by f_λ gives a one-to-one correspondence between left and right primitive ideals of $K[M]$ containing $cba - \lambda$.

We first describe all primitive ideals P of $K[M]$ such that $cba \in P$. We shall use the same notation for the elements of homomorphic images of $K[M]$ and the elements of $K[M]$, if unambiguous.

Theorem 3.5 *Let* P *be a left or right primitive ideal of the plactic algebra* $K[M]$ *of rank* 3 *over a field* K. *If* $cba \in P$ *then* P *is one of the following ideals*:

(1) *$(a-\alpha, b-\beta, c-\gamma)$ for $\alpha, \beta, \gamma \in K$ with $\alpha\beta\gamma = 0$,*
(2) *$(a, cb-\delta)$ or $(b, ca-\delta)$ or $(c, ba-\delta)$ for $0 \neq \delta \in K$.*

Conversely, each of these ideals is a left and right primitive ideal of $K[M]$.

Proof By Proposition 2.1 we have $cbK[M]a \subseteq P$. Hence $cb \in P$ or $a \in P$. In the former case, using the canonical form of elements in M, we get $cK[M]b \subseteq P$. Indeed, consider a word

$$w = (cba)^{k_1}(ba)^{k_2}(ca)^{k_3}(cb)^{k_4}a^{k_5}b^{k_6}c^{k_7} \in M,$$

where $k_i \geq 0$. If $k_1 > 0$ or $k_2 > 0$ or $k_4 > 0$ or $k_7 > 0$, then we have $cwb \in P$. Thus, we may assume $k_1 = k_2 = k_4 = k_7 = 0$. In this case, using $caa = aca$ and $cab = acb$, we have

$$cwb = c(ca)^{k_3}a^{k_5}b^{k_6}b = (ca)^{k_3}ca^{k_5}b^{k_6+1} = (ca)^{k_3}a^{k_5}cb^{k_6+1} \in P.$$

Hence $c \in P$ or $b \in P$. Summarizing, we have $a \in P$ or $b \in P$ or $c \in P$. Let us assume that $a \in P$ (the other cases can be considered similarly). Then $P/(a)$ may be viewed as a left or right primitive ideal of

$$K[M]/(a) \cong K[\langle b, c\rangle] \cong K[M_2].$$

Since the image of cb in $K[M]/(a)$ is central, Proposition 3.2 implies that $cb - \delta \in P$ for some $\delta \in K$.

In case $cb \in P$, i.e., $\delta = 0$, we have $c \in P$ or $b \in P$. In the latter case, $P/(a, b)$ may be viewed as a left or right primitive ideal of

$$K[M]/(a, b) \cong K[\langle c\rangle] \cong K[x],$$

the polynomial ring in one variable over K. Because $K[M]/(a, b)$ is commutative, $P/(a, b)$ is a maximal ideal, hence $c - \gamma \in P$ for some $\gamma \in K$. Thus we get $P = (a, b, c-\gamma)$.

In case $cb \notin P$, i.e., $\delta \neq 0$, $P/(a, cb-\delta)$ may be viewed as a left or right primitive ideal of

$$K[M]/(a, cb-\delta) \cong K[\langle b, c\rangle]/(cb-\delta) \cong K[B].$$

Thus, by Proposition 3.3, we have $P = (a, cb-\delta)$ or $P = (a, b-\beta, c-\gamma)$ for some $0 \neq \beta, \gamma \in K$ with $\beta\gamma = \delta$.

The remaining assertion of the theorem is obvious, because $K[B]$ is left and right primitive. This completes the proof of the theorem. □

Let P be a left primitive ideal of the algebra $K[M]$. In view of Theorem 3.5 we may restrict our attention to the case $cba \notin P$, so that, by Proposition 3.2, $cba - \lambda \in P$ for some $0 \neq \lambda \in K$. Moreover, Proposition 2.5 implies that $P_1 \subseteq P$ or $P_2 \subseteq P$.

We first construct some examples of simple left $K[M]$-modules with annihilators of this type.

Proposition 3.6 *Let V be a vector space over K with basis $\{e_{ij} : i, j \geq 0\}$. Let the action of $a, b, c \in M$ on V be given by*

$$ae_{ij} = e_{i,j+1}, \qquad be_{ij} = \begin{cases} \beta e_{ij} & \text{for } j = 0, \\ e_{i+1,j-1} & \text{for } j > 0, \end{cases} \qquad ce_{ij} = \begin{cases} 0 & \text{for } i = 0, \\ \gamma e_{i-1,j} & \text{for } i > 0, \end{cases}$$

where $\beta \in K$ and $0 \neq \gamma \in K$. Then this action makes V a simple left $K[M]$-module. Moreover, if P denotes the annihilator of V then

$$P = (ac - ca, (b - \beta)(acb - \gamma) + \beta(abc - bac), cba - \gamma).$$

Proof One can check that the given action on V respects the defining relations of M, so V is a left $K[M]$-module. Moreover, we have $e_{ij} = (ba)^i a^j e_{00}$ for $i, j \geq 0$, hence $V = K[M]e_{00}$. Therefore, to prove simplicity of V, it is enough to show that $e_{00} \in K[M]v$ for every $0 \neq v \in V$. Thus, let $0 \neq v \in V$. Then $c^{q+1}v = 0$ but $c^q v \neq 0$ for some $q \geq 0$. Replacing v by $c^q v \neq 0$ we may assume that $cv = 0$. In this case it is easy to see that $v = \sum_{j=0}^{t} \lambda_j e_{0j}$ for some $n \geq 0$ and $\lambda_j \in K$ with $\lambda_t \neq 0$. Now $e_{00} = (\gamma^t \lambda_t)^{-1}(cb)^t v \in K[M]v$. This completes the first part of the proof.

One can easily show that $ac - ca \in P$, $(b - \beta)(acb - \gamma) + \beta(abc - bac) \in P$ and $cba - \gamma \in P$.

First, note that if $\zeta \in K[\langle ba, a\rangle]$ satisfies $\zeta e_{00} = 0$ then $\zeta = 0$ in $K[M]$. Indeed, writing ζ in the form $\zeta = \sum_{i,j=0}^{r} \lambda_{ij}(ba)^i a^j$, where $\lambda_{ij} \in K$, we get

$$0 = \zeta e_{00} = \sum_{i,j=0}^{r} \lambda_{ij}(ba)^i a^j e_{00} = \sum_{i,j=0}^{r} \lambda_{ij} e_{ij},$$

hence $\lambda_{ij} = 0$ for all i, j and, in fact, $\zeta = 0$ in $K[M]$, as claimed. Moreover, observe that if $\omega \in K[\langle a, b\rangle]$ fulfills $\omega e_{0j} = 0$ for all $j \geq 0$, then $\omega = 0$ in $K[M]$. Indeed, let $\omega = \sum_{i=0}^{n} \omega_i b^i$, where $\omega_i \in K[\langle ba, a\rangle]$. Now, take $0 \leq d \leq n$ and suppose that $\omega_i = 0$ for $i < d$ (for $d = 0$ the condition is trivially fulfilled). Since

$$b^i e_{0d} = b^i a^d e_{00} = (ba)^d b^{i-d} e_{00} = \beta^{i-d}(ba)^d e_{00}$$

for $i \geq d$, we get

$$0 = \omega e_{0d} = \sum_{i=d}^{n} \omega_i b^i e_{0d} = (ba)^d \left(\sum_{i=d}^{n} \beta^{i-d} \omega_i \right) e_{00}.$$

Hence, by the previous observation, we obtain $(ba)^d \sum_{i=d}^{n} \beta^{i-d} \omega_i = 0$. This implies $\sum_{i=d}^{n} \beta^{i-d} \omega_i = 0$, because ba is a regular element of $K[\langle a, b\rangle] \cong K[M_2]$. If $d = n$, then the obtained equality yields $\omega_d = 0$, so we may assume $d < n$. Next, since

$$b^i e_{0,d+1} = b^i a^{d+1} e_{00} = \begin{cases} (ba)^d a e_{00} & \text{for } i = d, \\ \beta^{i-d-1}(ba)^{d+1} e_{00} & \text{for } i > d, \end{cases}$$

we have also

$$0=\beta\omega e_{0,d+1}=\sum_{i=d}^{n}\beta\omega_i b^i e_{0,d+1}=(ba)^d\left(\beta\omega_d a+\sum_{i=d+1}^{n}\beta^{i-d}\omega_i ba\right)e_{00},$$

hence $\beta\omega_d a+\sum_{i=d+1}^{n}\beta^{i-d}\omega_i ba=0$. Thus, since $\omega_d=-\sum_{i=d+1}^{n}\beta^{i-d}\omega_i$, we get

$$\beta\omega_d a=-\sum_{i=d+1}^{n}\beta^{i-d}\omega_i ba=\omega_d ba,$$

and Lemma 2.3 yields $\omega_d=0$. By induction on d, we conclude that $\omega=0$ in $K[M]$, as claimed.

Now, consider $\xi\in P$ as an element of

$$R=K[M]/(ac-ca,(b-\beta)(acb-\gamma)+\beta(abc-bac),cba-\gamma)$$

and assume, on the contrary, that $\xi\neq 0$. We shall use the fact that the elements of the set

$$E=\{a^{k_1}b^{k_2}(ba)^{k_3}c^{k_4},a^{k_1}b^{k_2}(cb)^{k_3}c^{k_4}:k_i\geq 0\}\subseteq R$$

constitute a basis of R over K (the proof of this technical fact is given in Lemma 3.7). Thus, we can write ξ in R, uniquely, as $\xi=\sum_{i,j=0}^{m}\tau_{ij}(ba)^i c^j+\sum_{i=1,\,j=0}^{m}\sigma_{ij}(cb)^i c^j$, where τ_{ij},σ_{ij} are linear combinations over K of the elements $a^k b^l$ for $k,l\geq 0$. Moreover, if $\tau_{ij}\neq 0$ or $\sigma_{ij}\neq 0$ for some $i\geq 0$ and $j\geq 2$, then

$$\xi ba=\sum_{i=0}^{m}\tau_{i0}(ba)^{i+1}+\gamma\sum_{i=0,\,j=1}^{m}\tau_{ij}(ba)^i c^{j-1}+\gamma\sum_{i=1}^{m}\sigma_{i0}b+\gamma\sum_{i,j=1}^{m}\sigma_{ij}(cb)^i c^{j-1}$$

is nonzero and the maximal exponent of c appearing in ξba is less than the maximal exponent in ξ. Similarly, if $\sigma_{ij}\neq 0$ for some $i\geq 2$ and $j\geq 0$, then

$$\xi a=\sum_{i,j=0}^{m}\tau_{ij}a(ba)^i c^j+\gamma\sum_{i=1,\,j=0}^{m}\sigma_{ij}(cb)^{i-1}c^j$$

is nonzero and the maximal exponent of cb appearing in ξa is less than in ξ. This allows us to assume that $\xi\neq 0$ has the form

$$\xi=\omega_0+\omega_1 c+\sigma_0 cb+\sigma_1 cbc,$$

where $\omega_i\in K[\langle a,b\rangle]$ and every σ_i is a linear combination over K of the elements $a^k b^l$ for $k,l\geq 0$. Since ξ, as an element of R, satisfies $\xi(ba)^{p+1}\in K[\langle a,b\rangle]$ for some $p\geq 0$ (in fact, $p=1$ is sufficient), we conclude, by the preceding part of the proof, that $\xi(ba)^{p+1}=0$ in R. Choosing p minimal, and replacing ξ by $\xi(ba)^p\neq 0$, we may assume $\xi ba=0$. Now, the equality

$$0=\xi ba=\omega_0 ba+\gamma\omega_1+\gamma\sigma_0 b+\gamma\sigma_1 cb$$

assures that $\sigma_1 = 0$ and $\omega_0 ba + \gamma\omega_1 + \gamma\sigma_0 b = 0$ in R. Furthermore, consider

$$\xi a = \omega_0 a + \omega_1 ac + \gamma\sigma_0.$$

Suppose, for a moment, that $\xi a \neq 0$. Then replacing ξ by ξa we reduce to the case where $\sigma_0 = 0$. Thus, we have $\xi = \omega_0 + \omega_1 c$. Moreover, in this case, the equality $0 = \xi ba = \omega_0 ba + \gamma\omega_1$ yields $0 \neq \gamma\xi = \omega_0(\gamma - bac)$ in R. Because $ce_{0j} = 0$ for all $j \geq 0$, we get

$$0 = \gamma\xi e_{0j} = \omega_0(\gamma - bac)e_{0j} = \gamma\omega_0 e_{0j}.$$

So, by the claim proved before, we conclude that $\omega_0 = 0$ and consequently, we get $\xi = 0$ in R. This contradiction implies that ξa must be equal to zero. Hence, in view of $0 = \xi a = \omega_0 a + \omega_1 ac + \gamma\sigma_0$, we obtain, in particular, $\omega_0 a + \gamma\sigma_0 = 0$ in R. Moreover, as noticed earlier, $\xi ba = 0$ implies $\omega_0 ba + \gamma\omega_1 + \gamma\sigma_0 b = 0$ in R. Therefore, these equalities yield

$$\gamma\sigma_0 = -\omega_0 a, \qquad \gamma\omega_1 = -\gamma\sigma_0 b - \omega_0 ba = \omega_0(ab - ba).$$

Thus we have

$$\begin{aligned}
\gamma\xi &= \gamma\omega_0 + \gamma\omega_1 c + \gamma\sigma_0 cb \\
&= \gamma\omega_0 + \omega_0(ab - ba)c - \omega_0 acb \\
&= \omega_0(\gamma - acb + abc - bac).
\end{aligned}$$

Furthermore

$$0 = \gamma\xi e_{00} = \omega_0(\gamma - acb + abc - bac)e_{00} = \gamma\omega_0 e_{00},$$

because $ce_{00} = cbe_{00} = 0$. Hence, we get $\omega_0 e_{00} = 0$. Now, if $\omega_0 = \sum_{i=0}^{s} \kappa_i b^i$, where $\kappa_i \in K[\langle ba, a\rangle]$, then

$$0 = \omega_0 e_{00} = \sum_{i=0}^{s} \kappa_i b^i e_{00} = \sum_{i=0}^{s} \beta^i \kappa_i e_{00}$$

implies that $\sum_{i=0}^{s} \beta^i \kappa_i = 0$ in R. Therefore, by subtracting $\sum_{i=0}^{s} \beta^i \kappa_i = 0$ from ω_0, we get $\omega_0 = \sum_{i=1}^{s} \kappa_i(b^i - \beta^i)$, and we conclude that $\omega_0 = \eta(b - \beta)$ for some $\eta \in K[\langle a, b\rangle]$. Finally, we have in R

$$\begin{aligned}
\gamma\xi &= \omega_0(\gamma - acb + abc - bac) \\
&= -\eta(b - \beta)(acb - \gamma + bac - abc) \\
&= -\eta((b - \beta)(acb - \gamma) + \beta(abc - bac)) = 0,
\end{aligned}$$

because $b(bac - abc) = 0$. This contradiction completes the proof. □

Lemma 3.7 *Let*

$$R = K[M]/(ac - ca, (b - \beta)(acb - \gamma) + \beta(abc - bac), cba - \gamma),$$

where $\beta \in K$ and $0 \neq \gamma \in K$. Then the set

$$E = \{a^{k_1}b^{k_2}(ba)^{k_3}c^{k_4}, a^{k_1}b^{k_2}(cb)^{k_3}c^{k_4} : k_i \geq 0\} \subseteq R$$

is a basis of R over K.

Proof The proof of this fact will be based on the Bergman's Diamond Lemma (cf. [1]). Consider the free algebra $K\langle x, y, z\rangle$ in three variables over K with the degree-lexicographic order in the corresponding free monoid $\langle x, y, z\rangle$ with $x < y < z$. Moreover, consider the set

$$F = \{x^{k_1}y^{k_2}(yx)^{k_3}z^{k_4}, x^{k_1}y^{k_2}(zy)^{k_3}z^{k_4} : k_i \geq 0\} \subseteq K\langle x, y, z\rangle,$$

and the reduction system S on $K\langle x, y, z\rangle$, given by the pairs:

$$\begin{aligned} &(yxx, xyx), \qquad (zyy, yzy), \qquad (zx, xz), \\ &(yyx, yxy), \qquad (zzy, zyz), \qquad (zyx, \gamma), \end{aligned}$$

and

$$(yxy^nzy, \gamma y^{n+1} + \beta^{n+1}(yxz + xzy - xyz) - \beta^{n+1}\gamma)$$

for $n \geq 0$. Note that, identifying the triple x, y, z in $K\langle x, y, z\rangle$ with a, b, c in R, all pairs in S come from relations holding in R. Indeed, it is obvious for the first six pairs and for $n = 0$ in the latter family of pairs listed above. Moreover, if $n \geq 0$, then by induction on n, we obtain in R

$$\begin{aligned} bab^{n+1}cb &= bbab^ncb \\ &= \gamma b^{n+2} + \beta^{n+1}(bbac + bacb + babc) - \beta^{n+1}\gamma b \\ &= \gamma b^{n+2} + \beta^{n+1}(bacb - \gamma b) \\ &= \gamma b^{n+2} + \beta^{n+2}(bac + acb - abc) - \beta^{n+2}\gamma. \end{aligned}$$

Further, since the elements:

$$yxx, \qquad yyx, \qquad zyy, \qquad zzy, \qquad zx, \qquad zyx, \qquad yxy^nzy$$

for $n \geq 0$, constitute the set of leading terms of pairs in the reduction system S, we have the following ambiguities of S:

$$y(yxx) = (yyx)x, \tag{1}$$

$$z(yxx) = (zyx)x, \tag{2}$$

$$z(yyx) = (zyy)x, \tag{3}$$

$$z(zyx) = (zzy)x, \tag{4}$$

$$z(zyy) = (zzy)y, \tag{5}$$

$$zy(yxx) = (zyy)xx, \tag{6}$$

$$zy(yyx) = (zyy)yx, \tag{7}$$

$$zz(yxx) = (zzy)xx, \tag{8}$$

$$zz(yyx) = (zzy)yx, \tag{9}$$

$$yxy^n(zyx) = (yxy^n zy)x, \tag{10}$$

$$yxy^n(zyy) = (yxy^n zy)y, \tag{11}$$

$$y(yxy^n zy) = (yyx)y^n zy, \tag{12}$$

$$z(yxy^n zy) = (zyx)y^n zy, \tag{13}$$

$$yxy^n z(yxx) = (yxy^n zy)xx, \tag{14}$$

$$yxy^n z(yyx) = (yxy^n zy)yx, \tag{15}$$

$$zy(yxy^n zy) = (zyy)xy^n zy, \tag{16}$$

$$zz(yxy^n zy) = (zzy)xy^n zy, \tag{17}$$

$$yxy^n z(yxy^m zy) = (yxy^n zy)xy^m zy \tag{18}$$

for $n, m \geq 0$. One may check that all the listed ambiguities are resolvable. Indeed, for example in case (6), (12) and (18) we have, respectively (some of the arrows indicate a sequence of reductions and not a single reduction):

$$\begin{aligned} zy(yxx) &\to zyxyx \to \gamma yx, \\ (zyy)xx &\to yzyxx \to \gamma yx \end{aligned}$$

and

$$\begin{aligned} y(yxy^n zy) &\to \gamma y^{n+2} + \beta^{n+1}(yyxz + yxzy - yxyz) - \beta^{n+1}\gamma y \\ &\to \gamma y^{n+2} + \beta^{n+1}(yxzy - \gamma y) \\ &\to \gamma y^{n+2} + \beta^{n+2}(yxz + xzy - xyz) - \beta^{n+2}\gamma, \\ (yyx)y^n zy &\to yxy^{n+1} zy \\ &\to \gamma y^{n+2} + \beta^{n+2}(yxz + xzy - xyz) - \beta^{n+2}\gamma \end{aligned}$$

and

$$\begin{aligned} yxy^n z(yxy^m zy) \to{}& \gamma yxy^n zy^{m+1} \\ &+ \beta^{m+1}(yxy^n zyxz + yxy^n zxzy - yxy^n zxyz) - \beta^{m+1}\gamma yxy^n z \\ \to{}& \gamma yxy^{n+m} zy, \\ (yxy^n zy)xy^m zy \to{}& \gamma y^{n+1} xy^m zy \\ &+ \beta^{n+1}(yxzxy^m zy + xzyxy^m zy - xyzxy^m zy) - \beta^{n+1}\gamma xy^m zy \\ \to{}& \gamma yxy^{n+m} zy. \end{aligned}$$

Similarly, one can check that all the other ambiguities of S are resolvable. Moreover, using Lemma 2.3 and the last two types of pairs in S, it is easy to see that F is equal to the set of all reduced, with respect to S, monomials in $K\langle x, y, z\rangle$. Hence the Diamond Lemma implies that the set F is a basis of $K\langle x, y, z\rangle/I$ over K, where I is the ideal of $K\langle x, y, z\rangle$ generated by the elements $w - f$ for all pairs $(w, f) \in S$ (cf. [1, Theorem 1.2]). It is clear that $R \cong K\langle x, y, z\rangle/I$ as K-algebras with the isomorphism given by $x, y, z \mapsto a, b, c$. Thus E, as an image of F under this isomorphism, is a basis of R over K. The assertion follows. □

Next, we construct another class of primitive ideals of $K[M]$ not containing the element cba.

Proposition 3.8 *Let V be a vector space over K with basis $\{e_i : i \geq 0\}$. Let the action of $a, b, c \in M$ on V be given by*

$$ae_i = \alpha e_i, \qquad be_i = \beta e_{i+1}, \qquad ce_i = \begin{cases} 0 & \text{for } i = 0, \\ \gamma e_{i-1} & \text{for } i > 0, \end{cases}$$

where $0 \neq \alpha, \beta, \gamma \in K$. Then this action makes V a simple left $K[M]$-module. Moreover, if P denotes the annihilator of V then $P = (a - \alpha, cb - \beta\gamma)$.

Proof One can check, as in Proposition 3.6, that V is a left $K[M]$-module. Moreover, if $v = \sum_{i=0}^{t} \lambda_i e_i \in V$, where $\lambda_i \in K$ and $\lambda_t \neq 0$ then $e_j = (\beta^j \gamma^t \lambda_t)^{-1} b^j c^t v$ for $j \geq 0$. This yields $V = K[M]v$, hence V is a simple $K[M]$-module.

Since $a - \alpha \in P$ and $cb - \beta\gamma \in P$, we have $(a - \alpha, cb - \beta\gamma) \subseteq P$. Consider $\xi \in P$ as an element of

$$R = K[M]/(a - \alpha, cb - \beta\gamma)$$

and suppose, on the contrary, that $\xi \neq 0$. One can check, as in Lemma 3.7, that the set $\{b^{k_1} c^{k_2} : k_i \geq 0\} \subseteq R$ is a basis of R over K. Thus, we can write ξ in R, uniquely, as $\xi = \sum_{i,j=0}^{m} \lambda_{ij} b^i c^j$, where $\lambda_{ij} \in K$. A similar argument as in Proposition 3.6 (used there, to decrease the maximal exponents of c and cb appearing in the form of ξ), based on right multiplication of ξ by b, allows us to assume that $\xi \neq 0$ has the form

$$\xi = \sum_{i=0}^{m} \mu_i b^i + \sum_{i=0}^{m} \nu_i b^i c,$$

where $\mu_i, \nu_i \in K$. Since ξ, as an element of R, satisfies $\xi b \in K[\langle b\rangle]$, it is easy to see that $\xi b = 0$ in R. Indeed, if $\zeta = \sum_{i=0}^{r} \lambda_i b^i$, where $\lambda_i \in K$, fulfills $\zeta e_0 = 0$, then

$$0 = \zeta e_0 = \sum_{i=0}^{r} \lambda_i b^i e_0 = \sum_{i=0}^{r} \beta^i \lambda_i e_i.$$

Hence $\lambda_i = 0$ for all i and, in fact, $\zeta = 0$ in R, as claimed. Using this observation, we get

$$0 = \xi b = \sum_{i=0}^{m} \mu_i b^{i+1} + \beta\gamma \sum_{i=0}^{m} \nu_i b^i$$

in R. This equality implies that $\mu_m = \nu_0 = 0$ and $\mu_{i-1} + \beta\gamma\nu_i = 0$ for all $i \neq 0$. Thus we get $\xi = \sum_{i=1}^{m} \nu_i b^{i-1}(bc - \beta\gamma)$ and

$$0 = \xi e_0 = -\beta\gamma \sum_{i=1}^{m} \nu_i b^{i-1} e_0 = -\beta\gamma \sum_{i=1}^{m} \beta^{i-1} \nu_i e_{i-1},$$

because $ce_0 = 0$. This yields $\nu_i = 0$ for all $i \neq 0$, so we conclude that $\xi = 0$. This contradiction completes the proof. □

As a consequence of the results obtained above, we get the following characterization of all left and right primitive ideals P of $K[M]$ such that $cba \notin P$.

Theorem 3.9 *Let P be a left or right primitive ideal of the plactic algebra $K[M]$ of rank 3 over a field K. If $cba \notin P$ then P is one of the following ideals*:

(1) $(a-\alpha, b-\beta, c-\gamma)$ *for* $0 \neq \alpha, \beta, \gamma \in K$,
(2) $(a-\alpha, cb-\delta)$ *or* $(c-\gamma, ba-\delta)$ *for* $0 \neq \alpha, \gamma, \delta \in K$,
(3) $(ac-ca, bacb-\lambda b, cba-\lambda)$ *for* $0 \neq \lambda \in K$,
(4) $(ac-ca, (b-\beta)(acb-\lambda)+\beta(abc-bac), cba-\lambda)$ *for* $0 \neq \beta, \lambda \in K$,
(5) $(bacb-\lambda b, (acb-\lambda)(ca-\beta)+\beta(\lambda^{-1}bacacb-bac), cba-\lambda)$ *for* $0 \neq \beta, \lambda \in K$.

Conversely, each of these ideals is a left and right primitive ideal of $K[M]$.

Proof Let P be a left primitive ideal of $K[M]$. By Proposition 3.2 we know that P contains $cba-\lambda$ for some $0 \neq \lambda \in K$. We start with the case where $P_1 \subseteq P$.

Let V be a simple left $K[M]$-module with the annihilator P. First, consider the case where

$$cv \neq 0 \quad \text{for every } 0 \neq v \in V. \tag{1}$$

Then c is invertible in $K[M]/P$ with inverse $\lambda^{-1}ba$. Hence c is central in $K[M]/P$, so, by Proposition 3.2, we have $c-\gamma \in P$ for some $0 \neq \gamma \in K$. This implies that $ba-\delta \in P$ for some $0 \neq \delta \in K$ with $\gamma\delta = \lambda$ and $P/(c-\gamma, ba-\delta)$ may be viewed as a left primitive ideal of

$$K[M]/(c-\gamma, ba-\delta) \cong K[\langle a, b\rangle]/(ba-\delta) \cong K[B].$$

Thus, by Proposition 3.3, we have $P = (c-\gamma, ba-\delta)$ or $P = (a-\alpha, b-\beta, c-\gamma)$ for some $0 \neq \alpha, \beta \in K$ with $\alpha\beta = \delta$.

Now, we assume $cv = 0$ for some $0 \neq v \in V$. If $(cb)^{p+1}v = 0$ but $(cb)^p v \neq 0$ for some $p \geq 0$, then we can replace v by $(cb)^p v \neq 0$, so we may assume

$$cv = cbv = 0 \quad \text{for some } 0 \neq v \in V. \tag{2}$$

Assuming (2) consider the case where $b^{q+1}v = 0$ but $b^q v \neq 0$ for some $q \geq 0$. Then we can replace v by $b^q v \neq 0$, so we may assume

$$bv = cv = 0 \quad \text{for some } 0 \neq v \in V. \tag{2a}$$

In this case Lemma 2.3 yields $V = \operatorname{Span}_K\{(ba)^i a^j v : i, j \geq 0\}$. Moreover, it is easy to see that the set $\{(ba)^i a^j v : i, j \geq 0\} \subseteq V$ is linearly independent over K. Indeed, suppose that we have a nontrivial relation $\sum_{i,j=0}^{t} \lambda_{ij}(ba)^i a^j v = 0$ for some $\lambda_{ij} \in K$. Let $i_0 = \max\{i : \lambda_{ij} \neq 0 \text{ for some } j\}$ and $j_0 = \max\{j : \lambda_{i_0 j} \neq 0\}$. Then we have

$$0 = (cb)^{j_0} c^{i_0} \sum_{i,j=0}^{t} \lambda_{ij}(ba)^i a^j v = \sum_{j=0}^{t} \lambda^{i_0} \lambda_{i_0 j}(cb)^{j_0} a^j v = \lambda^{i_0+j_0} \lambda_{i_0 j_0} v,$$

hence $\lambda_{i_0 j_0} = 0$, a contradiction. Therefore V is one of the modules constructed in Proposition 3.6 with the annihilator $P = (ac - ca, bacb - \lambda b, cba - \lambda)$.

In the case where

$$cv = cbv = 0 \quad \text{for some } 0 \neq v \in V \quad \text{and} \quad b^j v \neq 0 \quad \text{for every } j \geq 0, \tag{2b}$$

we have $\zeta bv = v$ for some $\zeta \in K[M]$, because V is a simple $K[M]$-module. Furthermore, in view of $cv = cbv = 0$ and Lemma 2.3, we may assume $\zeta \in K[\langle a, b\rangle]$. Thus we get $b^{k+1}\zeta \in K[\langle ba, b\rangle]$ for some $k \geq 0$. Since $b^{k+1}v \neq 0$ we have also $b^{k+1}\zeta \neq 0$. Writing $b^{k+1}\zeta$ in the form $b^{k+1}\zeta = \sum_{i,j=0}^{r} \lambda_{ij}(ba)^i b^j$, where $\lambda_{ij} \in K$, and defining $n = \max\{i : \lambda_{ij} \neq 0 \text{ for some } j\}$, we get

$$\sum_{j=0}^{r} \lambda^n \lambda_{nj} b^{j+1} v = c^n b^{k+1} \zeta bv = c^n b^{k+1} v = \begin{cases} b^{k+1}v & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Note that the equality displayed above yields a nontrivial relation of linear dependence. Indeed, if $n = 0$ and all λ_{0j} except λ_{0k} are equal to zero, then we get $b^{k+1}\zeta = \lambda_{0k} b^k$. This implies $\deg_b \zeta + k + 1 = k$, a contradiction. Similarly, if $n > 0$, then $\lambda_{nj} \neq 0$ for some j. Thus $b^{j+1}v$ is a linear combination of elements $b^i v$ for $i \neq j + 1$. This implies that b acts as an endomorphism of the finite-dimensional vector space $W = K[\langle b\rangle]v \subseteq V$ over K. Thus b has an eigenvector $0 \neq w \in W$, i.e., $bw = \beta w$ for some $\beta \in K$. Since $cv = cbv = 0$, we get also $cw = 0$ and $cbw = 0$. If $bw = 0$, then we are in case (2a). So, we may assume $\beta \neq 0$. In this case Lemma 2.3 yields $V = K[M]w = \operatorname{Span}_K\{(ba)^i a^j w : i, j \geq 0\}$. Moreover, as above, it is easy to see that the set $\{(ba)^i a^j w : i, j \geq 0\} \subseteq V$ is linearly independent over K. Hence V is one of the modules constructed in Proposition 3.6 with the annihilator $P = (ac - ca, (b - \beta)(acb - \lambda) + \beta(abc - bac), cba - \lambda)$.

In the case where

$$cv = 0 \quad \text{for some } 0 \neq v \in V \quad \text{and} \quad (cb)^j v \neq 0 \quad \text{for every } j \geq 0, \tag{3}$$

we have $\xi cbv = v$ for some $\xi \in K[M]$, because V is a simple $K[M]$-module. Furthermore, in view of $cv = 0$ and Lemma 2.3, we may assume $(cb)^{l+1}\xi \in K[\langle cb, b\rangle]$

for some $l \geq 0$. Moreover, since $(cb)^{l+1}v \neq 0$, we have also $(cb)^{l+1}\xi \neq 0$. Now, writing $(cb)^{l+1}\xi$ in the form $(cb)^{l+1}\xi = \sum_{i,j=0}^{s} \lambda_{ij}(cb)^i b^j$, where $\lambda_{ij} \in K$, and defining $m = \max\{j : \lambda_{ij} \neq 0 \text{ for some } i\}$, we get

$$\sum_{i=0}^{s} \lambda_{im}(cb)^{i+m+1}v = c^m(cb)^{l+1}\xi cbv = c^m(cb)^{l+1}v = \begin{cases} (cb)^{l+1}v & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

As before, one can show that the equality displayed above gives a nontrivial relation of linear dependence. This implies that cb acts as an endomorphism of the finite-dimensional vector space $W = K[\langle cb\rangle]v \subseteq V$ over K. Thus cb has an eigenvector $0 \neq w \in W$, i.e., $cbw = \delta w$ for some $\delta \in K$. Since $cv = 0$ we get also $cw = 0$, because $w \in W$. If $cbw = 0$ then we are in case (2). Hence, we may assume $\delta \neq 0$. Let $u = \delta aw - \lambda w$. Then we have $cu = 0$ and $cbu = 0$. If $u \neq 0$, then we are in case (2). If $u = 0$, then $cw = 0$, $cbw = \delta w$ and $aw = \alpha w$ with $\alpha\delta = \lambda$. Hence Lemma 2.3 yields $V = K[M]w = \mathrm{Span}_K\{(ba)^i w : i \geq 0\}$, because $bacbw = \lambda bw$. Moreover, it is easy to see that the set $\{(ba)^i w : i \geq 0\} \subseteq V$ is linearly independent over K. Hence V is one of the modules constructed in Proposition 3.8 with the annihilator $P = (a - \alpha, cb - \delta)$.

Now, consider the case where P is a left primitive ideal of $K[M]$ containing P_2. We still assume that $I_\lambda = (cba - \lambda) \subseteq P$ for some $0 \neq \lambda \in K$. Then, for f_λ from Lemma 3.4, $f_\lambda(P) + I_\lambda$ is a right primitive ideal of $K[M]$ containing P_1 and I_λ, because $f_\lambda(bacb - cbab) = \lambda^{-1}(cba)^2(ac - ca)$ and, in fact, $f_\lambda(P_2) + I_\lambda = P_1 + I_\lambda$. Therefore, for g from Lemma 3.4, $Q = g(f_\lambda(P)) + I_\lambda$ is a left primitive ideal of $K[M]$ containing P_1, because $g(P_1) = P_1$ and $g(I_\lambda) = I_\lambda$. Hence Q is one of the ideals of type (i), for some $i = 1, 2, 3, 4$, listed in the statement of the theorem. Since the sets of the ideals of type (i), for $i = 1, 2, 3, 4$, are invariant with respect to g, using $f_\lambda(cba - \lambda) = \lambda^{-1}((cba)^2 - \lambda^2)$ (in particular $f_\lambda(I_\lambda) \subseteq I_\lambda$) and the fact that $f_\lambda(f_\lambda(P)) + I_\lambda = P + I_\lambda$, we conclude that $P = f_\lambda(g(Q)) + I_\lambda$ is of type (j) for some $j = 1, 2, 3, 5$. Indeed, observe that, if Q is of type (i), for some $i = 1, 2, 3$, then P is also of type (i), whereas if Q is of type (4) then P is of type (5).

Finally, note that for every right primitive ideal P of $K[M]$ not containing cba, $Q = g(P)$ is a left primitive ideal of $K[M]$ and also does not contain cba. Hence Q is one of the ideals of type (k), for some $k = 1, 2, 3, 4, 5$, and $P = g(Q)$ is also of type (k), because each type is invariant with respect to g.

Moreover, each of the ideals of type (k) for $k = 1, 2, 3, 4, 5$ is a left and right primitive ideal of $K[M]$. This completes the proof of the theorem. □

Recall that a representation $K[M] \to \mathrm{End}_K(V)$ is said to be monomial, if there exists a basis E of V over K such that for every $w \in M$ and every $e \in E$ there exist $\lambda \in K$ and $f \in E$ satisfying $we = \lambda f$.

The following is now a direct consequence of Proposition 3.1, Theorem 3.5 and Theorem 3.9.

Corollary 3.10 *Every irreducible representation of $K[M]$ is monomial.*

Moreover, the ideals constructed in Theorem 3.5 and Theorem 3.9 are primitive ideals of $K[M]$ for any base field K. Indeed, Theorem 3.5 and Propositions 3.6, 3.8 remain true without our assumption that K is uncountable and algebraically closed.

Using families (4) and (5) of primitive ideals of $K[M]$ from Theorem 3.9, it is now possible to give a new simple proof of semiprimitivity of $K[M]$ (cf. [3]).

Proposition 3.11 *Let P_1 and P_2 be the minimal prime ideals of the plactic algebra $K[M]$ of rank 3 over an infinite field K. Then*

$$P_1 = \bigcap_{\beta,\lambda\in K\setminus\{0\}} (ac-ca, (b-\beta)(acb-\lambda)+\beta(abc-bac), cba-\lambda)$$

and similarly

$$P_2 = \bigcap_{\beta,\lambda\in K\setminus\{0\}} (bacb-\lambda b, (acb-\lambda)(ca-\beta)+\beta(\lambda^{-1}bacacb-bac), cba-\lambda).$$

Proof First, we establish the equality

$$P_1 = \bigcap_{\lambda\in K\setminus\{0\}} (ac-ca, cba-\lambda).$$

Since each of the ideals $(ac-ca, cba-\lambda)$ contains P_1, it is enough to show that $\bigcap_{\lambda\in K\setminus\{0\}}(cba-\lambda) = 0$ in $K[M]/P_1 \cong K[N_1]$. Now, fix $n \geq 1$. Then, for arbitrary pairwise distinct $0 \neq \lambda_1, \dots, \lambda_n \in K$, the ideals $I_j = (cba-\lambda_j)$ are pairwise comaximal in $K[N_1]$. Thus we get $\bigcap_{j=1}^n I_j = \prod_{j=1}^n I_j$, because cba is a central element of $K[N_1]$. This equality implies that every nonzero element $\zeta \in \bigcap_{j=1}^n I_j$ has the form $\zeta = \eta \prod_{j=1}^n (cba-\lambda_j)$ for some $0 \neq \eta \in K[N_1]$, hence we get $\deg \zeta \geq 3n$. Since n was chosen arbitrarily, we deduce that $\bigcap_{\lambda\in K\setminus\{0\}}(cba-\lambda)$ must be equal to zero in $K[N_1]$, as claimed.

Now, since each of the ideals $(ac-ca, (b-\beta)(acb-\lambda)+\beta(abc-bac), cba-\lambda)$ contains $(ac-ca, cba-\lambda)$, it suffices to show, for every $0 \neq \lambda \in K$, that

$$\bigcap_{\beta\in K\setminus\{0\}} ((b-\beta)(acb-\lambda)+\beta(abc-bac)) = 0$$

in

$$R = K[M]/(ac-ca, cba-\lambda) \cong K[N_1]/(cba-\lambda).$$

Therefore, let $0 \neq \lambda \in K$ be fixed. We know that, for every $0 \neq \beta \in K$, the action of the algebra R on a vector space $V^{(\beta)}$ over K with basis $\{e_{ij}^{(\beta)} : i, j \geq 0\}$, given in Proposition 3.6, makes $V^{(\beta)}$ a simple left R-module with the annihilator

$$P^{(\beta)} = ((b-\beta)(acb-\lambda)+\beta(abc-bac)) \subseteq R.$$

Moreover, let us observe that if $\omega \in K[\langle a, b\rangle]$ satisfies $\omega e_{00}^{(\beta)} = 0$ for all $0 \neq \beta \in K$, then $\omega = 0$ in R. Indeed, writing $\omega = \sum_{i,j,k=0}^r \lambda_{ijk}(ba)^i a^j b^k$, where $\lambda_{ijk} \in K$, we get

$$0 = \omega e_{00}^{(\beta)} = \sum_{i,j,k=0}^{r} \lambda_{ijk}(ba)^i a^j b^k e_{00}^{(\beta)} = \sum_{i,j,k=0}^{r} \beta^k \lambda_{ijk} e_{ij}^{(\beta)},$$

hence $\sum_{k=0}^{r} \beta^k \lambda_{ijk} = 0$ for all i, j. Then, by the Vandermonde argument, we conclude that $\lambda_{ijk} = 0$ for all i, j, k and, in fact, $\omega = 0$ in R, as claimed.

Now, consider $\xi \in \bigcap_{\beta \in K\setminus\{0\}} P^{(\beta)}$ as an element of R and assume, on the contrary, that $\xi \neq 0$. Then, as in Lemma 3.7, one may check that the elements of the set $\{(ba)^{k_1} a^{k_2} b^{k_3} (cb)^{k_4} c^{k_5} : k_i \geq 0\} \subseteq R$ constitute a basis of R over K. Thus we can write ξ in R, uniquely, as $\xi = \sum_{i,j=0}^{m} \omega_{ij}(cb)^i c^j$, where $\omega_{ij} \in K[\langle a, b\rangle]$. A similar argument as in Proposition 3.6 (used there, to decrease the maximal exponents of c and cb appearing in the form of ξ), based on right multiplication of ξ by ba and by a, allows us to assume that $\xi \neq 0$ has the form

$$\xi = \omega_{00} + \omega_{01}c + \omega_{10}cb + \omega_{11}cbc.$$

Since ξ, as an element of R, satisfies $\xi(ba)^{p+1} \in K[\langle a, b\rangle]$ for some $p \geq 0$ (in fact, $p = 1$ is sufficient), we conclude, by the preceding part of the proof, that $\xi(ba)^{p+1} = 0$ in R. Choosing p minimal and replacing ξ by $\xi(ba)^p \neq 0$ we may assume that $\xi ba = 0$. Moreover, exactly the same argument as in Proposition 3.6 (used there, to obtain a contradiction, if $\xi a \neq 0$) allows us to assume that $\xi a = 0$ and consequently, reduce to the situation where

$$\lambda\xi = \omega_{00}(\lambda - acb + abc - bac).$$

Therefore, since $cbe_{00}^{(\beta)} = ce_{00}^{(\beta)} = 0$, we get $0 = \xi e_{00}^{(\beta)} = \omega_{00}e_{00}^{(\beta)}$ for all $0 \neq \beta \in K$. Hence, by the previous paragraph of the proof, $\omega_{00} = 0$ and, in fact, $\xi = 0$. This contradiction completes the first part of the proof.

To complete the proof note that the second equality can by obtained from the first by using f_λ from Lemma 3.4. Indeed, for $0 \neq \lambda \in K$ we have

$$\begin{aligned}
&(bacb - \lambda b, (acb - \lambda)(ca - \beta) + \beta(\lambda^{-1}bacacb - bac), cba - \lambda) \\
&\quad = f_\lambda((ac - ca, (b - \beta)(acb - \lambda) + \beta(abc - bac), cba - \lambda)) + (cba - \lambda).
\end{aligned}$$

Therefore, applying f_λ to the equality

$$(ac - ca, cba - \lambda) = \bigcap_{\beta \in K\setminus\{0\}} (ac - ca, (b - \beta)(acb - \lambda) + \beta(abc - bac), cba - \lambda)$$

obtained before, we get

$$\begin{aligned}
&(bacb - \lambda b, cba - \lambda) \\
&\quad = f_\lambda((ac - ca, cba - \lambda)) + (cba - \lambda) \\
&\quad = \bigcap_{\beta \in K\setminus\{0\}} (bacb - \lambda b, (acb - \lambda)(ca - \beta) + \beta(\lambda^{-1}bacacb - bac), cba - \lambda).
\end{aligned}$$

So it is enough to show that $P_2 = \bigcap_{\lambda \in K\setminus\{0\}}(bacb - \lambda b, cba - \lambda)$. Since the proof of this equality is similar to the proof of $P_1 = \bigcap_{\lambda \in K\setminus\{0\}}(ac - ca, cba - \lambda)$, it will be omitted. The assertion follows. □

Corollary 3.12 *The plactic algebra $K[M]$ of rank 3 is semiprimitive for every base field K.*

Proof Let $\overline{K}$ be the algebraic closure of the field K. Then the field $\overline{K}$ is infinite. Hence, by Proposition 3.11, the algebra $\overline{K}[M]$ is semiprimitive. Therefore, by Amitsur Theorem (cf. [17, Theorem 7.2.13]), we conclude that the algebra $K[M]$ is also semiprimitive. □

Open Access This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

1. Bergman, G.M.: The diamond lemma for ring theory. Adv. Math. **29**, 178–218 (1978)
2. Cassaigne, J., Espie, M., Krob, D., Novelli, J.-C., Hivert, F.: The Chinese monoid. Int. J. Algebra Comput. **11**, 301–334 (2001)
3. Cedó, F., Okniński, J.: Plactic algebras. J. Algebra **274**, 97–117 (2004)
4. Cedó, F., Okniński, J.: Minimal spectrum and the radical of Chinese algebras. Preprint (2010)
5. Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups, vol. 1. Mathematical Surveys, vol. 7. American Mathematical Society, Providence (1977)
6. Duchamp, G., Krob, D.: Plactic-growth-like monoids. In: Words, Languages and Combinatorics II, pp. 124–142. World Scientific, Singapore (1994)
7. Fulton, W.: Young Tableaux. London Mathematical Society Student Texts, vol. 35. Cambridge University Press, New York (1997)
8. Gateva-Ivanova, T.: A combinatorial approach to set-theoretic solutions of the Yang-Baxter equation. J. Math. Phys. **45**, 3828–3858 (2004)
9. Jaszuńska, J., Okniński, J.: Chinese algebras of rank 3. Commun. Algebra **34**, 2745–2754 (2006)
10. Jaszuńska, J., Okniński, J.: Structure of Chinese algebras. J. Algebra (to appear), available at http://arxiv.org/abs/1009.5847v1
11. Jespers, E., Okniński, J.: Noetherian Semigroup Algebras. Algebra and Applications, vol. 7. Springer, Dordrecht (2007)
12. Krause, G.R., Lenagan, T.H.: Growth of Algebras and Gelfand-Kirillov Dimension. Graduate Studies in Mathematics, vol. 22. American Mathematical Society, Providence (2000)
13. Lam, T.Y.: A First Course in Noncommutative Rings. Graduate Texts in Mathematics, vol. 131. Springer, New York (2001)
14. Lascoux, A., Leclerc, B., Thibon, J.-Y.: The plactic monoid. In: Algebraic Combinatorics on Words, pp. 164–196. Cambridge University Press, Cambridge (2002)
15. Lascoux, A., Schützenberger, M.P.: Le monoïde plaxique. In: Noncommutative Structures in Algebra and Geometric Combinatorics, pp. 129–156. Dekker, Naples (1978)
16. Okniński, J.: Semigroup Algebras. Pure and Applied Mathematics, vol. 138. Dekker, New York (1991)
17. Passman, D.S.: The Algebraic Structure of Group Rings. Wiley-Interscience, New York (1977)